Elicitation of preference systems: Two procedures and their application to decision making under severe uncertainty

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ABSTRACT

In [1], we considered **preference systems** for modelling decision makers (DMs) with weakly structured preferences.

Now we present **elicitation procedures** enabling decision makers to report their underlying preference system while having to answer **fewest possible simple ranking questions**. Two different approaches are followed, based on either

- **consideration times** or
- **labels of preference strength**.

We briefly discuss applications to decision under uncertainty.

Quick tour: Only look at the gray boxes!

PRELIMINARIES

Let $A = \{a_1, \dots, a_n\}$ be a **finite set of consequences** that choices of a decision maker could possibly yield.

For a binary relation $R \subset A \times A$ on A, we denote by

- $P_R \subseteq A \times A$ its **strict part**
- $I_R \subseteq A \times A$ its **indifference part**
- $C_R \subseteq A \times A$ its **incomparable part**
- $H_R \subseteq A \times A$ its **transitive hull**

Definition 1 *A triplet $\mathcal{A} = [A, R_1, R_2]$ where $R_1 \subseteq A \times A$ is a pre-order on A and $R_2 \subseteq R_1 \times R_1$ is a pre-order on R_1 is called a* ***preference system*** *on A.*

Interpretation of preference systems:

- $(a, b) \in R_1$: "a at least as desirable as b" **(ordinal part)**
- $((a, b), (c, d)) \in R_2$: "exchanging b by a is at least as desirable as exchanging d by c" **(cardinal part)**

Definition 2 *$\mathcal{A} = [A, R_1, R_2]$ is called* ***consistent*** *if there exists a function $u : A \to [0, 1]$ such that for all $a, b, c, d \in A$:*

i) *$(a, b) \in R_1$ implies $u(a) \geq u(b)$ with equality iff $(a, b) \in I_{R_1}$.*

ii) *$((a, b), (c, d)) \in R_2$ implies $u(a) - u(b) \geq u(c) - u(d)$ with equality iff $((a, b), (c, d)) \in I_{R_2}$.*

The set of all such u is denoted by $\mathcal{U}_\mathcal{A}$.

Comment: Consistency can be checked by solving one linear optimization problem (see [1, Proposition 1]).

REFERENCES

[1] C. Jansen, G. Schollmeyer, and T. Augustin. Concepts for decision making under severe uncertainty with partial ordinal and partial cardinal preferences. *International Journal of Approximate Reasoning*, 98:112 – 131, 2018.

Further relevant references are given in [1].

MAIN QUESTION & IDEA

Goal: Elicit the decision maker's true preference system

$$\mathcal{A}^* = [A, R_1^*, R_2^*]$$

by asking as few as possible ranking questions about R_1^*.

Two different approaches:

A1: For every presented pair $\{a_i, a_j\}$ with $(a_i, a_j) \in R_1^*$, we measure the DM's **consideration time** $t_{ij} > 0$ and use these times for constructing R_2 (hopefully matching R_2^*).

A2: For every presented pair $\{a_i, a_j\}$ with $(a_i, a_j) \in R_1^*$, collect a label of **preference strength** and utilize the collected labels for constructing R_2 (hopefully matching R_2^*).

ASSUMPTIONS FOR A1

Intuition: The stonger the DM prefers a_i over a_j, the lower is the consideration time t_{ij}. More formal:

Assumption 1 *For $(a_i, a_j), (a_k, a_l) \in R_1^*$ the following holds:*

i) $t_{kl} > t_{ij} > 0$ if and only if $((a_i, a_j), (a_k, a_l)) \in P_{R_2^*}$

ii) $t_{kl} = t_{ij} > 0$ if and only if $((a_i, a_j), (a_k, a_l)) \in I_{R_2^*}$

iii) $t_{ij} = t_{ji} = c_\infty$ if and only if $(a_i, a_j) \in I_{R_1^*}$

Assumption 2 *For $(a_i, a_j), (a_j, a_k) \in P_{R_1^*}$ we have $\frac{1}{t_{ij}} + \frac{1}{t_{jk}} = \frac{1}{t_{ik}}$, whenever $(a_i, a_k) \in P_{R_1^*}$.*

Assumption 3 *For $(a_i, a_j) \in I_{R_1^*}$ we have*

i) $t_{ki} = t_{kj}$ whenever $(a_k, a_i), (a_k, a_j) \in P_{R_1^*}$ and

ii) $t_{ik} = t_{jk}$ whenever $(a_i, a_k), (a_j, a_k) \in P_{R_1^*}$.

ASSUMPTIONS FOR A2

Intuition: DM assigns a label $\ell_r^{ij} \in \mathcal{L}_r := \{\mathbf{n}, \mathbf{c}, 0, 1, \dots, r\}$ to every (a_i, a_j) by labelling function $\ell_r : A \times A \to \mathcal{L}_r$:

$\mathbf{n}$:	non-comparable
$\mathbf{c}$:	strict preference of unknown strength
0 :	indifferent
$1, \dots, r$:	strict preference of increasing strength

Assumption 4 *It holds that*

i) $(a_i, a_j) \in I_{R_1^*} \;\Leftrightarrow\; \ell_r^{ij} = 0$

ii) $(a_i, a_j) \in P_{R_1^*} \;\Leftrightarrow\; \ell_r^{ij} \in \mathcal{L}_r \setminus \{\mathbf{n}, 0\} \wedge \ell_r^{ji} = \mathbf{n}$

iii) $(a_i, a_j) \in C_{R_1^*} \;\Leftrightarrow\; \ell_r^{ij} = \ell_r^{ji} = \mathbf{n}$

Assumption 5 *For all $(a_i, a_j), (a_k, a_l) \in R_1^*$ the following holds:*

i) $\ell_r^{ij} > \ell_r^{kl} \;\Rightarrow\; ((a_i, a_j), (a_k, a_l)) \in P_{R_2^*}$

ii) $\ell_r^{ij} = \ell_r^{kl} = 0 \;\Rightarrow\; ((a_i, a_j), (a_k, a_l)) \in I_{R_2^*}$

iii) $\ell_r^{ij} = \mathbf{c} \vee \ell_r^{kl} = \mathbf{c} \;\Leftrightarrow\; ((a_i, a_j), (a_k, a_l)) \in C_{R_2^*}$

Assumption 6 *For all $((a_i, a_j), (a_k, a_l)) \in P_{R_2^*}$ the statement $\ell_r^{ij} = \ell_r^{kl} = x \notin \{0, \mathbf{n}, \mathbf{c}\}$ implies that $\{1, \dots, r\} \subset \ell_r(A \times A)$.*

A1: UTILIZING META DATA ON CONSIDERATION TIME

Procedure 1: Start with empty relations $R_1 = \emptyset$ and $C = \emptyset$ and ask the DM successively about the preferences between all pairs $\{a_i, a_j\} \in A_{\{2\}}$ of consequences, where $A_{\{2\}} := \{\{a, b\} : a \neq b \in A\}$. There are four possibilities:

i) DM judges a_i and a_j incomparable. Set $C = C \cup \{(a_j, a_i), (a_i, a_j)\}$ and $t_{ij} = t_{ji} = 0$.

ii) DM ranks a_i strictly better than a_j. Set $R_1 = R_1 \cup \{(a_i, a_j)\}$ and measure consideration time $t_{ij} > 0$. Set $t_{ji} = 0$.

iii) DM ranks a_j strictly better than a_i. Set $R_1 = R_1 \cup \{(a_j, a_i)\}$ and measure consideration time $t_{ji} > 0$. Set $t_{ij} = 0$.

iv) DM is indifferent between a_j and a_i. Set $R_1 = R_1 \cup \{(a_i, a_j), (a_j, a_i)\}$ and $t_{ij} = t_{ji} = c_\infty > \max\{t_{pq} : (a_p, a_q) \in P_{R_1^*}\}$.

Define R_2 by setting $((a_i, a_j), (a_k, a_l)) \in R_2 \quad :\Leftrightarrow \quad t_{kl} - t_{ij} \geq 0 \,\wedge\, t_{ij} > 0$.

Findings for Procedure 1:

Proposition 1 *Under Assumption 1, Procedure 1 produces the DM's true preference system $\mathcal{A}^* = [A, R_1^*, R_2^*]$.*

Proposition 2 *Under Assumptions 1, 2 and 3 the DM's true preference system $\mathcal{A}^* = [A, R_1^*, R_2^*]$ is consistent if and only if R_1^* is transitive.*

Procedure 1*: Suppose after k steps of Procedure 1 we have elicited R_1^k and C^k. Sample the next pair to present from

$$A_{\{2\}} \setminus \{\{a, b\} : (a, b) \in H_{R^k} \vee (b, a) \in H_{R^k} \vee (a, b) \in C^k\}$$

and *compute* the missing times by using Assumption 2.

Findings for Procedure 1*:

Proposition 3 *Under Assumptions 1, 2 and 3, Procedure 1* terminates in $\mathcal{A}^*$ if and only if R_1^* is transitive.*

A2: UTILIZING LABELS OF PREFERENCE STRENGTH

Procedure 2: Consider a DM with labelling function $\ell_r : A \times A \to \mathcal{L}_r$. Start with two empty relations $R_1 = \emptyset$ and $R_2 = \emptyset$ and ask the DM successively about the preferences between all pairs $(a_i, a_j) \in A \times A$. There are the following possibilities:

i) If $\ell_r^{ij} \in \mathcal{L}_r \setminus \{\mathbf{n}, 0\}$, set $R_1 = R_1 \cup \{(a_i, a_j)\}$.

ii) If $\ell_r^{ij} = 0$, set $R_1 = R_1 \cup \{(a_i, a_j), (a_j, a_i)\}$.

iii) If $\ell_r^{ij} = \mathbf{n}$, set $R_1 = R_1$.

Define R_2 by setting $((a_i, a_j), (a_k, a_l)) \in R_2 \quad :\Leftrightarrow \quad \ell_r^{ij} > \ell_r^{kl} \;\vee\; \ell_r^{ij} = \ell_r^{kl} = 0$

Findings for Procedure 2:

Proposition 4 *The following two statements hold true:*

i) If, for some $r \in \mathbb{N}$, $\ell_r : A \times A \to \mathcal{L}_r$ satisfies assumptions 4 and 5, then Procedure 2 produces a sub-system of the decision maker's true preference system $\mathcal{A}^*$. Particularly, the procedure produces a consistent preference system whenever $\mathcal{A}^*$ is consistent.

ii) There exists $r_0 \in \mathbb{N}$ such that if $\ell_{r_0} : A \times A \to \mathcal{L}_{r_0}$ satisfies assumptions 4, 5 and 6, then Procedure 2 produces the true $\mathcal{A}^*$.

OUTLOOK: DECISION MAKING UNDER SEVERE UNCERTAINTY

Situation: Suppose we have two uncertain acts $X_1, X_2 : S \to A$ and some probability π on S. To check if

$$(\star) \;\; \forall u \in \mathcal{U}_{\mathcal{A}^*} : \quad \mathbb{E}_\pi(u \circ X_1) \geq \mathbb{E}_\pi(u \circ X_2)$$

holds, we can proceed as follows: Denote by $\mathcal{A}_1, \mathcal{A}_2, \dots$ the preference system after step $1, 2, \dots$ of Procedure 1 or 2.

As this implies $\mathcal{U}_{\mathcal{A}_1} \supseteq \mathcal{U}_{\mathcal{A}_2} \supseteq \dots$, after each step k of the Procedure we can check (see [1, Prop. 5 i)]) if

$$\forall u \in \mathcal{U}_{\mathcal{A}_k} : \quad \mathbb{E}_\pi(u \circ X_1) \geq \mathbb{E}_\pi(u \circ X_2)$$

and directly conclude $(\star)$ if so.

DM solves problem by only answering simple ranking questions about R_1^*!