Decision making with state-dependent preference systems

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Abstract. In this paper we present some first ideas for decision making with agents whose preference system may depend on an uncertain state of nature. Our main formal framework here are commonly scalable state-dependent decision systems. After giving a formal definition of those systems, we introduce and discuss two criteria for defining optimality of acts, both of which are direct generalizations of classical decision criteria under risk. Further, we show how our criteria can be naturally extended to imprecise probability models. More precisely, we consider convex and finitely generated credal sets. Afterwards, we propose linear pogramming-based algorithms for evaluating our criteria and show how the complexity of these algorithms can be reduced by approximations based on clustering the preference systems under similar states. Finally, we demonstrate our methods in a toy example.

Keywords: state-dependent preferences · preference system · imprecise probabilities · decision making under uncertainty · linear programming.

1 Introduction

In many applications, the agent's preferences in a decision making problem under uncertainty can not be modeled independently of the true *state of nature*. Prominent examples for such applications can, e.g., be found in the field of insurance science. Here, often a policyholder's preferences are modelled to be dependent on her health status as, e.g., being reliant on the help of other people may lead to different preferences as being completely autonomous (see [3] for a very recent work in this direction). Further examples can, e.g., be found in problems of portfolio selection, where commonly the agent's attitude towards risky choices (and therefore indirectly the underlying preferences) are seen as depending on some exogenous environment (see, e.g., [22] for a recent work).

In such situations, the decision maker's preferences are called *state-dependent*: The knowledge of the true state of nature might force the decision maker to completely (or partially) rearrange the ranking of the consequences different decisions may lead to. Given their practical relevance, it is not surprising that many fundamental works have dealt with state-dependent preferences. For instance, one can consult the classic sources [8] or [15, 16], but many more exist. See also [2] for a modern reappraisal. Most of these works are in the classical Anscombe and

Aumann framework of ‘preferences over horse lotteries’ (see [1]): Starting from a preference relation on the domain of all horse lotteries, they derive ‘conditional’ preference relations for every state fixed and then say the original relation is state-dependent whenever there exist distinct (non-null) states for which these conditional relations differ (see, e.g., [9]).

In the present work, we choose a more direct and applied view on the notion of state dependence. Instead of over horse lotteries, we model preferences directly on a finite consequence set and assume that the uncertainty about the states is externally given by an imprecise probability model. Moreover, under each state we allow the agent to express also partial preferences with respect to both the ordering itself and the strength of preferences. The practical evaluation of consequences often relies on reference points external to the consequence itself (see, e.g., [7] for impactful psychological research), and indeed these reference points may also be related to other consequences. A quite prominent example for such a setting is obtained by rigorously formalizing the notion of regret familiar from classical decision theory: measuring the ”inappropriateness” of an action in a particular state, as [14, p.59] originally had put it, is impossible ”unless [...] state-contingent consequences can be specified” ([12, p. 810]), see also, for instance, [17] on axiomatisations of the minimax regret principle and, e.g., [13] for a recent application in the context of climate model uncertainty.

The paper is organized as follows: Section 2 discusses the required mathematical definitions and concepts. After that, Section 3 introduces the notion of state-dependent decision systems and proposes two classes of decision crtieria for both the case of precise and imprecise probabilistic information about the states. In Section 4, we first demonstrate how the proposed criteria can be evaluated by using linear optimization theory and then discuss different possibilities for reducing the complexity of the obtained linear programs by grouping the variables under ‘similar’ states. Section 5 illustrates the discussed concepts in a toy example. Section 6 concludes the paper.

2 Preliminaries

We start by recalling our central concept for modelling a decision maker’s preferences, namely the concept of a *preference system* as introduced in [6]. The basic idea here is very natural: the ordinal and the cardinal part of the preferences are modeled by two separate pre-orders (i.e. transitive and reflexive binary relations). The ordinal order is a pre-order on the set of consequences, while the cardinal order formally corresponds to a pre-order on the ordinal order - conceived as a set. Note that the following Definitions 1, 2 and 3 are (essentially) taken from [6].

Definition 1. *Let A be a non-empty set and let $R_1 \subseteq A \times A$ denote a pre-order on A. Moreover, let $R_2 \subseteq R_1 \times R_1$ denote a pre-order on R_1. Then the triplet $\mathcal{A} = [A, R_1, R_2]$ is called a* ***preference system*** *on A. The preference system $\mathcal{A}' = [A, R_1', R_2']$ is called* ***sub-system*** *of $\mathcal{A}$ if $R_1' \subseteq R_1$ and $R_2' \subseteq R_2$.*

To ensure that the two orders of a preference system are compatible and do not contradict, a consistency criterion is introduced. Roughly speaking, a preference system is consistent if there exists a utility function on the set of consequences which represents both involved pre-orders simultaneously. As usual, we will use the following notation: For a pre-order $R \subseteq M \times M$ on a set M, we denote by $P_R \subseteq M \times M$ its *strict part*[1] and by $I_R \subseteq M \times M$ its *indifference part*[2].

Definition 2. *Let $\mathcal{A} = [A, R_1, R_2]$ be a preference system. Then $\mathcal{A}$ is said to be* ***consistent*** *if there exists a function $u : A \to [0,1]$ such that for all $a, b, c, d \in A$ the following properties hold:*

i) If $(a,b) \in R_1$, then $u(a) \geq u(b)$ with equality iff $(a,b) \in I_{R_1}$.
ii) If $((a,b),(c,d)) \in R_2$, then

$$u(a) - u(b) \geq u(c) - u(d)$$

with equality iff $((a,b),(c,d)) \in I_{R_2}$.

Every such function u is then said to ***(weakly) represent*** *the preference system $\mathcal{A}$. The set of all (weak) representations u of $\mathcal{A}$ is denoted by $\mathcal{U}_{\mathcal{A}}$.*

For consistent preference systems whose ordinal order has minimal and maximal elements, it may be useful to consider only utility functions that measure the utility of consequences on the same scale. This is, for example, central for defining an expected value generalized to preference systems.

Definition 3. *Let $\mathcal{A} = [A, R_1, R_2]$ be a consistent preference system. Assume there exist elements $a_*, a^* \in A$ such that $(a^*, a) \in R_1$ and $(a, a_*) \in R_1$ for all $a \in A$. Then the set*

$$\mathcal{N}_{\mathcal{A}} := \Big\{ u \in \mathcal{U}_{\mathcal{A}} : u(a_*) = 0 \ \wedge \ u(a^*) = 1 \Big\}$$

is called the ***normalized representation set*** *of $\mathcal{A}$. Further, for a number $\delta \in [0,1)$, we denote by $\mathcal{N}_{\mathcal{A}}^{\delta}$ the set of all $u \in \mathcal{N}_{\mathcal{A}}$ satisfying*

$$u(a) - u(b) \geq \delta \quad \wedge \quad u(c) - u(d) - u(e) + u(f) \geq \delta$$

for all $(a,b) \in P_{R_1}$ and for all $((c,d),(e,f)) \in P_{R_2}$. Then, $\mathcal{N}_{\mathcal{A}}^{\delta}$ is called the ***normalized representation set of granularity*** *δ of $\mathcal{A}$.*

3 State-dependent decision systems

We now come to the central concept of this paper, namely that of state-dependent decision systems. The idea is very natural: Instead of a fixed preference system, we now want to allow the decision maker's preferences to be dynamic in the states of nature of a decision problem under uncertainty.

[1] Defined by: $(m_1, m_2) \in P_R \Leftrightarrow (m_1, m_2) \in R \wedge (m_2, m_1) \notin R$
[2] Defined by: $(m_1, m_2) \in I_R \Leftrightarrow (m_1, m_2) \in R \wedge (m_2, m_1) \in R$

3.1 The basic model

We start by giving the fundamental definition. Note that for simplicity and to avoid measure-theoretic problems, the sets $A = \{a_1, \dots, a_n\}$ and $S = \{s_1, \dots, s_m\}$ are assumed to be finite throughout the rest of the paper.

Definition 4. *Let S denote some non-empty set of states of nature and A denote some non-empty set of consequences. For every $s \in S$, let $\mathcal{A}_s = [A, R_1^s, R_2^s]$ be a preference system on A. For a non-empty subset $\mathcal{G} \subseteq A^S := \{f : S \to A\}$, we call the pair*

$$\mathcal{D} = \left[\mathcal{G}, \left(\mathcal{A}_s\right)_{s \in S}\right]$$

a ***decision system****. A decision system $\mathcal{D}$ is called* ***state-independent*** *if it holds that $\mathcal{A}_s = \mathcal{A}_{s'}$ for all $s, s' \in S$. Otherwise, $\mathcal{D}$ will be called* ***state-dependent****.*

Especially in the case of a state-dependent decision system, it is useful to consider only utility functions that measure the utility on the same scale. In contrast to non-dynamic preference systems, however, this requires a stronger assumption: The maximal and minimal elements of the ordinal orders of all preference systems involved must be independent of the state of the nature.

Definition 5. *Let $\mathcal{D} = \left[\mathcal{G}, (\mathcal{A}_s)_{s \in S}\right]$ be a decision system. Then $\mathcal{D}$ is called* ***commonly scalable*** *if there exist elements $a_*, a^* \in A$ such that $(a^*, a) \in R_1^s$ and $(a, a_*) \in R_1^s$ for all $a \in A$ and $s \in S$, i.e. if there exist common maximal and minimal elements which are independent of the state of nature. Further, for $\delta \in [0, 1)$, $\mathcal{D}$ is called δ-**consistent** if $\mathcal{N}_{\mathcal{A}_s}^{\delta} \neq \emptyset$ for all $s \in S$. Finally, $\mathcal{D}$ is called* ***consistent*** *if it is 0-consistent.*

Note that the definition of a state-dependent commonly scalable decision system $\mathcal{D} = \left[\mathcal{G}, (\mathcal{A}_s)_{s \in S}\right]$ does not rule out the possibility that there are states under which the decision maker has coinciding preference systems. From now on we assume, *without restricting generality of what follows*, that for some $\ell \in \{1, \dots m\}$ there is a partition $\mathbb{S} := \{S_1, \dots, S_\ell\}$ of S that satisfies the following properties:

i) For all $d \in \{1, \dots, \ell\}$ and all $s_{i_1}, s_{i_2} \in S_d$ it holds $\mathcal{A}_{s_{i_1}} = \mathcal{A}_{s_{i_2}}$.
ii) For all $c \neq d \in \{1, \dots, \ell\}$ and all $s_{i_1} \in S_c$ and $s_{i_2} \in S_d$ it holds $\mathcal{A}_{s_{i_1}} \neq \mathcal{A}_{s_{i_2}}$.
iii) For $c < d \in \{1, \dots, \ell\}$, if $s_{i_1} \in S_c$ and $s_{i_2} \in S_d$, then $i_1 < i_2$.

We then denote by $\mathcal{A}_{S_d}$ the preference system $\mathcal{A}_s$ for arbitrary $s \in S_d$. Note that this assumption simply ensures that the states of nature are already grouped in classes containing coinciding preference systems.

3.2 Criteria for decision making

We will now consider two different types of decision criteria in state-dependent decision systems: Criteria based on numerical representations of a generalized expected value and those that select undominated elements of a generalized stochastic dominance relation. For the first type of criterion, we first need to define what is meant by expected value in our context.

Definition 6. *Let $\mathcal{D} = \left[\mathcal{G}, (\mathcal{A}_s)_{s\in S}\right]$ be a commonly scalable and δ-consistent decision system and let π denote a probability measure on $(S, 2^S)$. For $X \in \mathcal{G}$, we define the expressions*

$$L_\pi^\delta(X) = \min\left\{\sum_{d=1}^{\ell}\left(\sum_{s\in S_d} u_{S_d}(X(s)) \cdot \pi(\{s\})\right) : (u_{S_1}, \dots, u_{S_\ell}) \in \bigtimes_{d=1}^{\ell} \mathcal{N}_{\mathcal{A}_{S_d}}^\delta\right\}$$

$$U_\pi^\delta(X) = \max\left\{\sum_{d=1}^{\ell}\left(\sum_{s\in S_d} u_{S_d}(X(s)) \cdot \pi(\{s\})\right) : (u_{S_1}, \dots, u_{S_\ell}) \in \bigtimes_{d=1}^{\ell} \mathcal{N}_{\mathcal{A}_{S_d}}^\delta\right\}$$

Then, the (possibly degenerated) interval

$$E_\pi^\delta(X) = [L_\pi^\delta(X), U_\pi^\delta(X)]$$

*is called the **state-dependent expectation** of X with respect to the prior distribution π and granularity δ.*

In principle, there are many different criteria thinkable that are based on the state-dependent expectations $E_\pi^\delta(X)$ of the different acts in $X \in \mathcal{G}$. Here, we want to stick with the most conservative one among them, namely the one taking into account only the lower bound $L_\pi^\delta(X)$ of each of these intervals.

Definition 7. *Let $\mathcal{D} = \left[\mathcal{G}, (\mathcal{A}_s)_{s\in S}\right]$ be a commonly scalable and δ-consistent decision system and let π denote a probability measure on $(S, 2^S)$. An act $X^* \in \mathcal{G}$ is called $(\mathcal{D}, \pi, \delta)$**-maximin** if $L_\pi^\delta(X^*) \geq L_\pi^\delta(X)$ for all $X \in \mathcal{G}$.*

Some remarks on Definition 7: in the case of a state-independent decision system, the criterion essentially reduces to the $\mathcal{D}_\delta$-maximin criterion as introduced in [6, Definition 6 i)]. If further the then constant relations R_1 and R_2 satisfy the axioms in [11, p. 147, Definition 1] and thus admit a cardinal utility representation that is unique up to positive linear transformations, the criterion reduces to the principle of maximizing expected utility (note that of course in this case it is implied that both relations R_1 and R_2 have to be complete).

The second type of criterion is based on a generalization of the concept of first order stochastic dominance. The idea is first to define a partial order on the set of available acts and then to call optimal those among them that are undominated with respect to this relation. We start with defining the dominance relation.

Definition 8. *Let $\mathcal{D} = \left[\mathcal{G}, (\mathcal{A}_s)_{s\in S}\right]$ be a commonly scalable and consistent decision system and let π denote a probability measure on $(S, 2^S)$. For an act $X \in \mathcal{G}$ and a collection of functions $u := (u_d)_{d=1,\dots,\ell}$ such that it holds that $u_d \in \mathcal{N}_{\mathcal{A}_{S_d}}$ for each $d = 1, \dots, \ell$, the expression*

$$E_{(\pi,u)}(X) = \sum_{d=1}^{\ell}\left(\sum_{s\in S_d} u_{S_d}(X(s)) \cdot \pi(\{s\})\right)$$

*is called the (π, u)**-expectation** of X. Further, for two acts $X, Y \in \mathcal{G}$, we say that X $(\mathcal{D}, \pi)$**-dominates** Y, abbreviated with $X \geq_{(\mathcal{D},\pi)} Y$, if it holds that*

$$E_{(\pi,u)}(X) \geq E_{(\pi,u)}(X)$$

for every $u := (u_d)_{d=1,\dots,\ell}$ *with* $u_d \in \mathcal{N}_{\mathcal{A}_{S_d}}$ *for each* $d = 1, \dots, \ell$*. Finally, an act* $X^* \in \mathcal{G}$ *is called* $(\mathcal{D}, \pi)$***-undominated*** *if there is no act* $Y \in \mathcal{G}$ *such that* $Y \geq_{(\mathcal{D},\pi)} X^*$ *but not* $X^* \geq_{(\mathcal{D},\pi)} Y$*, that is if* X^* *is an undominated element of the dominance relation* $\geq_{(\mathcal{D},\pi)}$*.*

Some remarks on the dominance relation: First, it is immediate that for a state-independent decision system in which, in addition, the cardinal relation R_2 involved is empty, classical stochastic dominance for partial orders is equivalent to it. Further, it can be easily shown that for the case of a state-independent decision system with constant, but not necessarily empty, R_2, it reduces to the order $R_{\forall\forall}$ as defined in [6, p. 123] and also considered in [5, Definition 4 ii)]. Finally, one sees immediately that the dominance relation is a pre-order on $\mathcal{G}$, i.e., a reflexive and transitive, but not necessarily complete, binary relation.

3.3 Generalizing the criteria to imprecise probabilities

So far, we have limited our considerations to decision systems under precise probabilities. In this section we want to show how the decision criteria discussed so far can also be generalized most naturally to decision systems under imprecise probabilities. Although different generalizations are often conceivable (analogous as in the case with cardinal utility, see, e.g., [21, 18]), we restrict ourselves to one particular generalization each for reasons of simplicity and space. As a generalized uncertainty model, we consider convex and finitely generated credal sets $\mathcal{M}$, i.e., convex sets of probability measures on $(S, 2^S)$ with a finite number of extreme points collected in

$$\mathcal{E}(\mathcal{M}) = \{\pi^{(1)}, \dots \pi^{(K)}\}.$$

We start by generalizing $(\mathcal{D}, \pi, \delta)$-maximin. As already mentioned, many different generalizations are plausible, depending on the decision maker's attitude towards ambiguity. Consistent with our previous restriction to the lower bound, we will again examine only the absolute ambiguity-averse variant.

Definition 9. *Let* $\mathcal{D} = \left[\mathcal{G}, (\mathcal{A}_s)_{s\in S}\right]$ *be a commonly scalable and* δ*-consistent decision system and let* $\mathcal{M}$ *be a convex and finitely generated credal set on* $(S, 2^S)$*. An act* $X^* \in \mathcal{G}$ *is called* $(\mathcal{D}, \mathcal{M}, \delta)$***-maximin*** *if*

$$L^{\delta}_{(\mathcal{D},\mathcal{M})}(X^*) := \min_{\pi\in\mathcal{M}} L^{\delta}_{\pi}(X^*) \geq \min_{\pi\in\mathcal{M}} L^{\delta}_{\pi}(X) =: L^{\delta}_{(\mathcal{D},\mathcal{M})}(X)$$

for all $X \in \mathcal{G}$*.*

This decision criterion also has a well-known special case: If the underlying decision system is state-independent and the then constant relations of the preference system guarantee a unique utility representation up to for positive linear transformations (see above), then the criterion reduces to the Γ-maximin criterion known from decision making under imprecise probabilities (see, e.g., [18]).

Also the dominance relation $\geq_{(\mathcal{D},\pi)}$ from Definition 8 can naturally be adapted to the case of imprecise probabilities by demanding the involved acts to be in relation for all probability measures from the underlying credal set.

Definition 10. *Let $\mathcal{D} = \left[\mathcal{G}, (\mathcal{A}_s)_{s\in S}\right]$ be a commonly scalable and consistent decision system and let $\mathcal{M}$ be a convex and finitely generated credal set on $(S, 2^S)$. For acts $X, Y \in \mathcal{G}$, we say X $(\mathcal{D},\mathcal{M})$**-dominates** Y, abbreviated with $X \geq_{(\mathcal{D},\mathcal{M})} Y$, if it holds that $X \geq_{(\mathcal{D},\pi)} Y$ for every $\pi \in \mathcal{M}$. Further, an act $X^* \in \mathcal{G}$ is called $(\mathcal{D},\mathcal{M})$**-undominated** if X^* is $(\mathcal{D},\pi)$-undominated for every $\pi \in \mathcal{M}$.*

4 Algorithms for determining optimal acts

In this section we show how optimal acts can be determined with respect to the discussed decision criteria using linear optimization as has been extensively done before in the context of decision making with imprecise probabilities (see, e.g., [20, 10, 4, 19]). For this purpose, we first discuss two basic algorithms for the optimization of the two criteria discussed and then demonstrate how the complexity of these algorithms can be reduced by suitable approximations. The idea of the approximation is to group the preference systems under certain, in a certain sense similar, states and then to consider only decision variables for each cluster of states in the optimization. All discussed optimization problems are given directly for the criteria under imprecise probabilities, since these contain the criteria under precise probabilities in each case as a special case.

4.1 Two basic linear programs

We start with the basic linear program for computing the criterion value of any fixed act with respect to the $(\mathcal{D},\mathcal{M},\delta)$-maximin criterion. For this let $\mathcal{A} = [A, R_1, R_2]$, with $A = \{a_1, \ldots, a_n\}$, be a consistent preference system for which there exist elements $a_{k_1}, a_{k_2} \in A$ such that $(a_{k_1}, a) \in R_1$ and $(a, a_{k_2}) \in R_1$ for all $a \in A$. The property of a vector $(\alpha_1, \ldots, \alpha_n)$ to contain exactly the images of a utility function $u : A \to [0, 1]$ from the set $\mathcal{N}_{\mathcal{A}}^{\delta}$ is then describable via a system of linear (in-)equalities.

More precisely, next to the equalities $u_{k_1} = 1$ and $u_{k_2} = 0$, for every pair $(a_i, a_j) \in R_1$ we receive the linear inequality $u_i - u_j \geq \delta$ and for every pair of pairs $((a_k, a_l), (a_p, a_q)) \in R_2$, we receive the linear inequality $u_k - u_l - u_p + u_q \geq \delta$. Denote by $\Delta_{\mathcal{A}}^{\delta}$ the set of all vectors $(\alpha_1, \ldots, \alpha_n) \in [0, 1]^n$ satisfying all these (in)equalities. Equipped with this, we receive the following proposition which can be proved by slightly adapting the proof of [6, Prop. 3].

Proposition 1. *Let $\mathcal{D} = \left[\mathcal{G}, (\mathcal{A}_s)_{s\in S}\right]$ be a commonly scalable and δ-consistent decision system with common maximal and minimal elements $a_{k_1}, a_{k_2} \in A$, respectively, and let $A = \{a_1, \ldots, a_n\}$ and $S = \{s_1, \ldots s_m\}$. Let $\mathcal{M}$ be a convex and finitely generated credal set on $(S, 2^S)$. For $X \in \mathcal{G}$, denote by w_j the unique $i \in \{1, \ldots, n\}$ with $X(s_j) = a_i$. For every fixed $t \in \{1, \ldots, K\}$, consider the linear program*

$$\sum_{d=0}^{\ell-1} \left(\sum_{j=c_d+1}^{c_{d+1}} v_{w_j}^{d} \cdot \pi^{(t)}(\{s_j\}) \right) \longrightarrow \min_{(v_1^1,\ldots,v_n^1,\ldots,v_1^\ell,\ldots v_n^\ell)}$$

with constraints $(v_1^1, \dots, v_n^1, \dots, v_1^\ell, \dots v_n^\ell) \in \Delta^\delta_{\mathcal{A}_{S_d}}$ *for every* $d \in \{1, \dots, \ell\}$ *and the conventions* $S_0 = \emptyset$ *and* $c_d = |\cup_{j=0}^d S_j|$*. Denote by* $opt(t)$ *the optimal value of the linear program with* t *fixed. It then holds:*

$$L^\delta_{(\mathcal{D},\mathcal{M})}(X) = \min\Big\{ opt(t) : t \in \{1, \dots, K\} \Big\}$$

We now turn to the basic linear program for checking $(\mathcal{D}, \mathcal{M})$-dominance. The proposition can be proven by slightly modifying the proof of [6, Prop. 5 i)].

Proposition 2. *Consider the same situation as in Proposition 1. For* $X, Y \in \mathcal{G}$*, denote by* x_j *and* y_j *the unique* $i_X, i_Y \in \{1, \dots, n\}$ *such that* $X(s_j) = a_{i_X}$ *and* $Y(s_j) = a_{i_Y}$ *hold, respectively. For every fixed* $t \in \{1, \dots, K\}$*, consider the linear optimization problem*

$$\sum_{d=0}^{\ell-1} \left(\sum_{j=c_d+1}^{c_{d+1}} (v^d_{x_j} - v^d_{y_j}) \cdot \pi^{(t)}(\{s_j\}) \right) \longrightarrow \min_{(v_1^1, \dots, v_n^1, \dots, v_1^\ell, \dots v_n^\ell)}$$

with constraints $(v_1^1, \dots, v_n^1, \dots, v_1^\ell, \dots v_n^\ell) \in \Delta^0_{\mathcal{A}_{S_d}}$ *for every* $j \in \{1, \dots, m\}$ *and the conventions* $S_0 = \emptyset$ *and* $c_d = |\cup_{j=0}^d S_j|$*. Denote by* $opt(t)$ *the optimal value of the linear program with* t *fixed. It then holds:*

$$X \geq_{(\mathcal{D},\mathcal{M})} Y \quad \Leftrightarrow \quad \min\Big\{ opt(t) : t \in \{1, \dots, K\} \Big\} \geq 0$$

We end the paragraph with two brief comments: First, to check whether an act X is undominated, Proposition 2 can simply be applied several times: If $Y \geq_{(\mathcal{D},\mathcal{M})} X$ does *not* hold for all acts $Y \in \mathcal{G} \setminus \{X\}$, then one can directly infer the undominatedness of X. Second, both propositions can also be applied to precise probability measures. In this case, one simply chooses the credal set $\mathcal{M} = \{\pi\}$ as a singleton consisting only of the precise probability in question. The propositions then simplify considerably, since in each case only one instead of a set of linear programs has to be solved.

4.2 Approximating the linear programs by grouping the states

The linear programs from Propositions 1 and 2 possess separate variables and constraints for the preference system under every state of nature of the decision system. This may produce very complex optimization tasks if the considered decision problem is large. In this section, we will look at how to significantly reduce both the number of variables and the number of constraints without sacrificing too much accuracy. The main idea is to approximate the discussed basic algorithms by grouping the preference systems under in a certain sense similar states of nature. However, before turning to the approximations just mentioned, let us first note a fundamental property of preference and, as a consequence, also decision systems. It follows by observing that the intersection of pre-orders is again a pre-order that preserves minimal and maximal elements.

Proposition 3. *Let $\mathcal{D} = \big[\mathcal{G}, (\mathcal{A}_s)_{s\in S}\big]$ be a commonly scalable decision system with state space S and let $\mathbb{V}$ be some partition of S. Then it holds that*

$$\mathcal{D}_{\mathbb{V}} := \big[\mathcal{G}, (\mathcal{A}_s^{\mathbb{V}})_{s\in S}\big]$$

is a commonly scalable decision system, where for every $V \in \mathbb{V}$ and $s \in V$:

$$\mathcal{A}_s^{\mathbb{V}} := \left[A, \bigcap\nolimits_{d\in V} R_1^d, \bigcap\nolimits_{d\in V} R_2^d\right].$$

Note that this implies $\mathcal{A}_{s_1}^{\mathbb{V}} = \mathcal{A}_{s_2}^{\mathbb{V}}$ for $s_1, s_2 \in V \in \mathbb{V}$, i.e. the preference systems are constant within the partition classes.

In what follows we are interested in partitions $\mathbb{V}$ of the state space S of which the partition $\mathbb{S}$ already discussed is a *refinement*: For every element $S_d \in \mathbb{S}$ there exists an element $V \in \mathbb{V}$ such that $S_d \subseteq V$. We denote this by $\mathbb{S}\#\mathbb{V}$ and also call $\mathbb{V}$ a *coarsening* of $\mathbb{S}$ in this case.

Let us assume that we already found a suitable partition $\mathbb{V} = \{V_1, \ldots, V_r\}$ of the state space, where $r \leq \ell$ and $\mathbb{S}\#\mathbb{V}$. Similar as already done for $\mathbb{S}$, we assume without loss of generality that for $c < d \in \{1, \ldots, r\}$, if $s_{i_1} \in V_c$ and $s_{i_2} \in V_d$, then $i_1 < i_2$. The idea to approximate our basic algorithms by less complex, but preferably information-preserving surrogate algorithms is then very simple: Instead of considering separate variables and constraints for each preference system under each state, we only consider separate variables and constraints for each element of the partition provided with the common preference system.

Technically, this is achieved by replacing the series of linear programming problems from Proposition 1 by the series of problems (for every $t \in \{1, \ldots, K\}$)

$$\sum_{d=0}^{r-1}\left(\sum_{j=p_d+1}^{p_{d+1}} v_{w_j}^d \cdot \pi^{(t)}(\{s_j\})\right) \longrightarrow \min_{(v_1^1,\ldots,v_n^1,\ldots,v_1^r,\ldots v_n^r)}$$

with constraints $(v_1^1, \ldots, v_n^1, \ldots, v_1^r, \ldots v_n^r) \in \Delta^{\delta}_{\mathcal{A}^{\mathbb{V}}_{V_d}}$ for every $d \in \{1, \ldots, r\}$ and the conventions $V_0 = \emptyset$ and $p_d = |\cup_{j=0}^{d} V_j|$ and by, respectively, replacing the series of linear programming problems from Proposition 2 by the series of linear programming problems (for every $t \in \{1, \ldots, K\}$)

$$\sum_{d=0}^{\ell-1}\left(\sum_{j=p_d+1}^{p_{d+1}} (v_{x_j}^d - v_{y_j}^d) \cdot \pi^{(t)}(\{s_j\})\right) \longrightarrow \min_{(v_1^1,\ldots,v_n^1,\ldots,v_1^r,\ldots v_n^r)}$$

with constraints $(v_1^1, \ldots, v_n^1, \ldots, v_1^r, \ldots v_n^r) \in \Delta^{0}_{\mathcal{A}^{\mathbb{V}}_{V_d}}$ for every $d \in \{1, \ldots, r\}$ and, again, $V_0 = \emptyset$ and $p_d = |\cup_{j=0}^{d} V_j|$. As the approximation quality heavily depends on it, the partition should be chosen in an information-preserving manner.

4.3 Different choices for the partition

So far, we have restricted the partition $\mathbb{V}$ of the state space only in that it had to be a coarsening of the partition $\mathbb{S}$. In this section, we will now briefly discuss two concrete choices for $\mathbb{V}$ that have other desirable properties besides this minimal requirement. For that, assume that $(\mathcal{A}_s)_{s\in S}$ denotes the family of preference systems corresponding to a (potentially state-dependent) decision system.

Pattern clustering: The first possibility for a partition of the state space is to group preference systems that contain a certain predefined preference pattern. Let $\mathcal{P}_1, \dots, \mathcal{P}_z$, where $z < \ell$, denote pairwisely conflicting[3] preference systems on A. Then, a partition $\mathbb{V}_{pa}$ in at most $z+1$ partition classes is obtained by

$$\mathbb{V}_{pa} = \Big\{\{s \in S : \mathcal{P}_t \preceq \mathcal{A}_s\} : t = 1, \dots, z\Big\} \cup \Big\{\{s \in S : \mathcal{P}_t \not\preceq \mathcal{A}_s \text{ for all } t\}\Big\},$$

where for preference systems $\mathcal{B} = [A, R_1^B, R_2^B]$ and $\mathcal{C} = [A, R_1^C, R_2^C]$ we denote by $\mathcal{B} \preceq \mathcal{C}$ that $P_{R_1^B} \subseteq P_{R_1^C}$, $I_{R_1^B} \subseteq I_{R_1^C}$, $P_{R_2^B} \subseteq P_{R_2^C}$ and $I_{R_2^B} \subseteq I_{R_2^C}$ hold.

Distance-based clustering: Another possibility for finding a partition of the state space is to group states $s \in S$ whose associated ordinal relations R_1^s are not 'too far' away from each other: As described in [23, Algorithm 1], for some distance d between pre-orders (like, e.g., the normalized cardinality of their symmetric difference), one first picks a threshold $\xi \in (0,1)$ and computes the distances $d(R_1^s, R_1^{s^*})$ for all $s \neq s^* \in S$. Afterwards, we put such states in the same cluster between which there exists a 'path of ordinal relations' with distances lower or equal than ξ. This gives a partition of S into some number of clusters $C_1, \dots, C_b$. If one now extends the distance function to clusters by setting $D(C_{l_1}, C_{l_2}) := \min\{d(R_1^s, R_1^{s^*}) : s \in C_{l_1}, s^* \in C_{l_2}\}$, one can repeat this step until the partition does no longer change.

5 An illustrative toy example

As an illustrative example, we consider the simple commonly scalable decision system given in Table 1 with only two acts taking values in the consequence set $A = \{a_*, b, c, d, a^*\}$, where a_* and a^* denote the common minimal and maximal elements of A, respectively. Under each state $s \in \{s_1, s_2, s_3\}$, we assume that R_1^s

	$\mathbf{s_1}$	$\mathbf{s_2}$	$\mathbf{s_3}$
$\mathbf{X_1}$	d	c	b
$\mathbf{X_2}$	a^*	d	a_*

Table 1. A compact representation of the decision system.

is given by the transitive hull of of the chain $a^* P_{R_1^s} d P_{R_1^s} c P_{R_1^s} b P_{R_1^s} a_*$. Thus, the ordinal part of the preferences is state independent. In contrast, the cardinal part of the preferences does depend on the state of nature: For $a_1, a_2 \in A$, denote

[3] For $d_1 \neq d_2 \in \{1, \dots, z\}$ we have $\mathcal{U}_{\mathcal{P}_{d_1}} \cap \mathcal{U}_{\mathcal{P}_{d_2}} = \emptyset$. This makes $\mathbb{V}_{pa}$ a partition.

by $e_{a_1a_2}$ the pair (a_1, a_2). Then, under s_1 the cardinal part $R_2^{s_1}$ is given as the transitive hull of $e_{ba_*}I_{R_2^{s_1}}e_{cb}I_{R_2^{s_1}}e_{dc}I_{R_2^{s_1}}e_{a^*d}$, under s_2 the cardinal part $R_2^{s_2}$ is given as the transitive hull of $e_{ba_*}P_{R_2^{s_2}}e_{cb}P_{R_2^{s_2}}e_{a^*d}P_{R_2^{s_2}}e_{dc}$ and under s_3 the cardinal part $R_2^{s_3}$ is given as the transitive hull of $e_{ba_*}P_{R_2^{s_3}}e_{a^*d}P_{R_2^{s_3}}e_{cb}P_{R_2^{s_3}}e_{dc}$.

We make the following three observations: (1) The preference system $\mathcal{A}_{s_1}$ *uniquely* specifies a utility function $u_{s_1} \in \mathcal{N}_{\mathcal{A}_{s_1}}$, which is given by

$$(u_{s_1}(a_*), u_{s_1}(b), u_{s_1}(c), u_{s_1}(d), u_{s_1}(a^*)) = (0, 0.25, 0.5, 0.75, 1).$$

(2) The preference system $\mathcal{A}_{s_2}$ restricts all utility functions $u_{s_2} \in \mathcal{N}_{\mathcal{A}_{s_2}}$ to satisfy the inequality $u_{s_2}(d) - u_{s_2}(c) \leq 0.25$. (3) The preference system $\mathcal{A}_{s_3}$ restricts all utility functions $u_{s_3} \in \mathcal{N}_{\mathcal{A}_{s_3}}$ to satisfy the inequality $u_{s_3}(b) - u_{s_3}(a_*) \geq 0.25$.

Now, assume the uncertainty about the states of nature is characterized by the credal set $\mathcal{M} = \big\{\pi : \pi(\{s_1\}) \leq 0.2 \ \wedge \ \pi(\{s_2\}) \leq 0.2\big\}$. Then, for arbitrary $\pi \in \mathcal{M}$, $u_{s_1} \in \mathcal{N}_{\mathcal{A}_{s_1}}$, $u_{s_2} \in \mathcal{N}_{\mathcal{A}_{s_2}}$ and $u_{s_3} \in \mathcal{N}_{\mathcal{A}_{s_3}}$ the expression $E_{(\pi,u)}(X_1) - E_{(\pi,u)}(X_2)$ can be computed by

$$-\underbrace{\pi_1(u_{s_1}(a^*) - u_{s_1}(d))}_{\leq 0.2 \cdot 0.25} - \underbrace{\pi_2(u_{s_2}(d) - u_{s_2}(c))}_{\leq 0.2 \cdot 0.25} + \underbrace{\pi_3(u_{s_3}(b) - u_{s_3}(a_*))}_{\geq 0.6 \cdot 0.25} > 0.$$

Thus, as the probability and the utility are arbitrary, this inequality demonstrates that $X_1 \geq_{(\mathcal{D},\mathcal{M})} X_2$. An approximation under distance-based clustering yields the same: Since the ordinal relations are state-independent, any distance produces only one cluster $\{S\}$. Thus, we obtain a state-independent decision system and the intersection of $R_2^{s_1}$, $R_2^{s_2}$ and $R_2^{s_3}$ contains (e_{ba_*}, e_{dc}) and (e_{ba_*}, e_{a^*d}). Thus, the above inequality still holds since $\pi_1 + \pi_2 < \pi_3$ for all $\pi \in \mathcal{M}$.

6 Outlook

In this paper, we have presented some initial ideas on decision theory with state-dependent preference systems. Besides the conceptual foundation and the differentiation from existing notions of state-dependence, we have focused on the computation of the presented decision criteria. We proposed different linear programs and showed how they can be approximated by less complex ones if the states are clustered appropriately. While we think that our paper gives a solid formal basis for decision making with state-dependent preference systems, we are also aware there is still a lot left open for future research. Example directions could be to systematically investigate which cluster techniques work best and to apply the concepts developed in this paper to real world data.

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